THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4030 Differential Geometry
3 October, 2024 Tutorial Solutions

1. (*Easy warm-up*, From exercise 2-5.1 of [**docarmo**]) Compute the first fundamental forms of the following parameterised surfaces where they are regular:

   (a) Ellipsoid $X(u_1, u_2) = (a \sin u_1 \cos u_2, b \sin u_1 \sin u_2, c \cos u_1),\ a, b, c$ constants

   (b) Hyperbolic paraboloid $X(u_1, u_2) = (au_1 \cosh u_2, bu_1 \sinh u_2, u_1^2)$, a, b, c constants

Solution. (a) We have

$$\begin{aligned} X_1 &= (a \cos u_1 \cos u_2, b \cos u_1 \sin u_2, -c \sin u_1) \\ X_2 &= (-a \sin u_1 \sin u_2, b \sin u_1 \cos u_2, 0) \end{aligned}$$

and so

$$\begin{aligned} g_{11} &= X_1 \cdot X_1 \\ &= a^2 \cos^2 u_1 \cos^2 u_2 + b^2 \cos^2 u_1 \sin^2 u_2 + c^2 \sin^2 u_1 \\ &= (a^2 \cos^2 u_2 + b^2 \sin^2 u_2) \cos^2 u_1 + c^2 \sin^2 u_1, \end{aligned}$$

$$\begin{aligned} g_{21} = g_{12} &= X_1 \cdot X_2 \\ &= (b^2 - a^2) \cos u_1 \sin u_1 \cos u_2 \sin u_2, \end{aligned}$$

and finally

$$\begin{aligned} g_{22} &= X_2 \cdot X_2 \\ &= (a^2 \sin^2 u_2 + b^2 \cos^2 u_2) \sin^2 u_1 \end{aligned}$$

so we have

$$g = \begin{pmatrix} (a^2 \cos^2 u_2 + b^2 \sin^2 u_2) \cos^2 u_1 + c^2 \sin^2 u_1 & (b^2 - a^2) \cos u_1 \sin u_1 \cos u_2 \sin u_2 \\ (b^2 - a^2) \cos u_1 \sin u_1 \cos u_2 \sin u_2 & (a^2 \sin^2 u_2 + b^2 \cos^2 u_2) \sin^2 u_1 \end{pmatrix}.$$

(b) We have

$$\begin{aligned} X_1 &= (a \cosh u_2, b \sinh u_2, 2u_1) \\ X_2 &= (au_1 \sinh u_2, bu_1 \cosh u_2, 0) \end{aligned}$$

and so after computation, we find

$$g = \begin{pmatrix} a^2 \cosh^2 u_2 + b^2 \sinh^2 u_2 + 4u_1^2 & (a^2 + b^2)u_1 \cosh u_2 \sinh u_2 \\ (a^2 + b^2)u_1 \cosh u_2 \sinh u_2 & u_1^2(a^2 \sinh^2 u_2 + b^2 \cosh^2 u_2) \end{pmatrix}.$$

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2. (From exercise 2-5.3 of [**docarmo**]) Find the parameterisation of the unit sphere $\mathbb{S}^2$ using stereographic projection from the north pole $N = (0, 0, 1)$ and find the coefficients of the first fundamental form with respect to the stereographic projection of the unit sphere.

Solution. Deriving stereographic projection from the north pole. Let $(u, v) \in \mathbb{R}^2$ and let P be the line connecting (u, v) to N. Note that we are projecting onto the $z = 0$ plane so $(u, v) \in \mathbb{R}^2$ has coordinates in $\mathbb{R}^3$ given by $(u, v, 0)$. We parameterise P by

$$P(t) = (0, 0, 1) + t(u, v, -1)$$

so that at $t = 1$, $P(1) = (u, v, 0)$. Note that we could have also chosen to project onto the $z = -1$ plane and P would need to be adjusted accordingly.

We solve for t so that $P(t)$ lies on $\mathbb{S}^2$, i.e. $|P(t)|^2 = 1$. We have

$$(tu)^2 + (tv)^2 + (1 - t)^2 = 1 \Leftrightarrow t = \frac{2}{1 + u^2 + v^2}.$$

Then with this value for t, we find

$$P(t) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)$$

so we find that stereographic projection is given by

$$X(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right)$$

and one can check that the conditions for a regular surface are satisfied by X.

We compute the first fundamental form:

$$X_1 = \left(\frac{-2u^2 + 2v^2 + 2}{(1 + u^2 + v^2)^2}, \frac{-4uv}{(1 + u^2 + v^2)^2}, \frac{4u}{(1 + u^2 + v^2)^2} \right),$$
$$X_2 = \left(\frac{-4uv}{(1 + u^2 + v^2)^2}, \frac{2u^2 - 2v^2 + 2}{(1 + u^2 + v^2)^2}, \frac{4v}{(1 + u^2 + v^2)^2} \right)$$

and, after computation, we have

$$g_{11} = \langle X_1, X_1 \rangle = \frac{4}{(1 + u^2 + v^2)^2}$$
$$g_{12} = \langle X_1, X_2 \rangle = 0 = \langle X_2, X_1 \rangle = g_{21}$$
$$g_{22} = \langle X_2, X_2 \rangle = \frac{4}{(1 + u^2 + v^2)^2}$$

that is,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \frac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4}{(1+u^2+v^2)^2} \end{pmatrix}.$$

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3. Consider the sphere parameterised by spherical coordinates:

$$X(u,v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

with $u \in (-\pi, \pi)$ and $v \in (0, \pi)$. Find the length of the curve γ given by $u = u_0$ and $a \leq v \leq b$ with $0 < a < b < \pi$.

Solution. We first compute the coefficients of the first fundamental form. We have

$$X_u = (-\sin v \sin u, \sin v \cos u, 0),$$
$$X_v = (\cos v \cos u, \cos v \sin u, -\sin v),$$

and

$$g = \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\gamma(t) = X(\gamma_1(t), \gamma_2(t)) = X(u_0, t)$, we see that $\gamma_1'(t) = 0$ and $\gamma_2'(t) = 1$. Using the first fundamental form, we find that the length of γ is given by

$$\begin{aligned} L\left(\gamma|_{[a,b]}\right) &= \int_a^b \left(\sum_{i,j=1}^{2} g_{ij}(\gamma(t)) \cdot \gamma_i'(t)\gamma_j'(t) \right)^{\frac{1}{2}} dt \\ &= \int_a^b \left(\sin^2 v \cdot 0^2 + 0 \cdot 0 \cdot 1 + 0 \cdot 1 \cdot 0 + 1 \cdot 1^2\right)^{\frac{1}{2}} dt \\ &= \int_a^b dt = b - a. \end{aligned}$$

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4. (*Time permitting*, Exercise 2-5.14 of [**docarmo**]) The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map $\operatorname{grad}(f) : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector $\operatorname{grad}(f)_p \in T_pS$ so that for all $v \in T_pS$,

$$\operatorname{grad}(f)_p \cdot v = df_p(v).$$

   (a) Express $\operatorname{grad}(f)$ in terms of the coefficients of the first fundamental form and the partial derivatives of f on the local parameterisation $X : U \to S$ at $p \in X(U)$.

   (b) Let $p \in S$ and $\operatorname{grad}(f)_p \neq 0$. Show that $v \in T_pS$ with $|v| = 1$ satisfies

$$df_p(v) = \max\{df_p(u) : u \in T_p(S), |u| = 1\}$$

   if and only if $v = \dfrac{\operatorname{grad}(f)_p}{|\operatorname{grad}(f)_p|}$. (*Thus, grad*$(f)_p$ *gives the direction of maximum variation of* f *at* p.)

Solution. Let S be parameterised by $X(u_1, u_2)$ at p. We want to write $\operatorname{grad}(f)_p$ in the basis $\{X_1(p), X_2(p)\}$, that is, find constants $\alpha, \beta \in \mathbb{R}$ such that

$$\operatorname{grad}(f)_p = \alpha X_1 + \beta X_2.$$

By the property characterising $\text{grad}(f)_p$ given above, we have that

$$\text{grad}(f)_p \cdot X_1 = df_p(X_1)$$

where on the right hand side we have

$$df_p(X_1) = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_1$$

where f_1 denotes $\frac{\partial f}{\partial u_1}$. On the left hand side, we have

$$\text{grad}(f)_p \cdot X_1 = (\alpha X_1 + \beta X_2) \cdot X_1 = \alpha g_{11} + \beta g_{12}.$$

So we have that

$$f_1 = \alpha g_{11} + \beta g_{12}.$$

Similarly, taking dot product with X_2, we find that

$$f_2 = \alpha g_{12} + \beta g_{22}$$

using the fact that dot product is symmetric and hence $g_{21} = g_{12}$. So we arrive at the system of equations

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

and solving this system for α, β yields

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

that is,

$$\alpha = \frac{f_1 g_{22} - f_2 g_{12}}{g_{11}g_{22} - g_{12}^2}$$
$$\beta = \frac{f_2 g_{11} - f_1 g_{12}}{g_{11}g_{22} - g_{12}^2}$$

so

$$\text{grad}(f)_p = \frac{f_1 g_{22} - f_2 g_{12}}{g_{11}g_{22} - g_{12}^2} X_1 + \frac{f_2 g_{11} - f_1 g_{12}}{g_{11}g_{22} - g_{12}^2} X_2.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} df_p(v) &= \text{grad}(f)_p \cdot v \\ &\leq |\text{grad}(f)_p \cdot v| \\ &\leq |\text{grad}(f)_p||v| = |\text{grad}(f)_p| \end{aligned}$$

when v is a unit vector, and the maximum is attained if and only if v and $\text{grad}(f)_p$ are parallel, that is, if $v = \lambda \text{grad}(f)_p$ for some $\lambda > 0$. Since v is a unit vector, this forces $\lambda = \dfrac{1}{|\text{grad}(f)_p|}$. ◀